

A GROUPOID STRUCTURE ON A VECTOR SPACE

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Abstract. In this paper we introduce the concept of generalized vector groupoid. Several properties of them are established. ¹

1 Introduction

The notion of groupoid was introduced by H. Brandt [Math. Ann., **96**(1926), 360-366; MR 1512323]. This algebraic structure is similar to a group, with the exception that products of elements cannot be always defined.

A generalization of Brandt groupoid has appeared in a paper of C. Ehresmann [Oeuvres completes. Parties I.1, I.2. Dunod, Paris, 1950]. Groupoids and their generalizations (topological groupoids, Lie and symplectic groupoids etc.) are mathematical structures that have proved to be useful in many areas of science (see for instance [1], [2], [3], [5], [7], [8]).

The concept of vector groupoid has been defined by V. PopuȚa and Gh. Ivan [6]. In this paper we introduce a groupoid structure in the sense of Ehresmann on a vector space.

The paper is organized as follows. In Section 2 we present some basic facts about Ehresmann groupoids. In Section 3 we introduce the notion of generalized vector groupoids and its useful properties are established. The construction of the induced vector groupoid and some characterizations of them are given in Section 4.

2 Ehresmann groupoids

We recall the minimal necessary backgrounds on Ehresmann groupoids (for further details see e.g. [5], [8]).

Definition 2.1. ([5]) A **groupoid** G over G_0 (in the sense of Ehresmann) is a pair (G, G_0) of sets endowed with two surjective maps $\alpha, \beta : G \rightarrow G_0$ (called **source**, respectively **target**), a partially binary operation (called **multiplication**) $m : G_{(2)} := \{(x, y) \in G \times G \mid \beta(x) = \alpha(y)\} \rightarrow G$, $(x, y) \mapsto m(x, y) := x \cdot y$, ($G_{(2)}$ is the **set of composable pairs**), an injective map $\varepsilon : G_0 \rightarrow G$ (called **inclusion map**) and a map $i : G \rightarrow G$, $x \mapsto i(x) := x^{-1}$ (called **inversion**), which verify the following conditions:

(G1) (**associativity**): $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ in the sense that if one of two products $(x \cdot y) \cdot z$ and $x \cdot (y \cdot z)$ is defined, then the other product is also defined and they are equals;

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(G2) **(units)**: for each $x \in G \implies (\varepsilon(\alpha(x)), x) \in G_{(2)}, (x, \varepsilon(\beta(x))) \in G_{(2)}$ and $\varepsilon(\alpha(x)) \cdot x = x = x \cdot \varepsilon(\beta(x))$;
 (G3) **(inverses)**: for each $x \in G \implies (x^{-1}, x) \in G_{(2)}, (x, x^{-1}) \in G_{(2)}$ and $x^{-1}x = \varepsilon(\beta(x)), xx^{-1} = \varepsilon(\alpha(x))$.

For a groupoid G we sometimes write $(G, \alpha, \beta, m, \varepsilon, \iota, G_0)$ or (G, α, β, G_0) or (G, G_0) ; G_0 is called the *base* of G . The functions $\alpha, \beta, m, \varepsilon, i$ are called the *structure functions* of G . The element $\varepsilon(\alpha(x))$ respectively $\varepsilon(\beta(x))$ is called the *left unit* respectively *right unit* of x ; $\varepsilon(G_0)$ is called the *unit set* of G . For each $u \in G_0$, the set $\alpha^{-1}(u)$ (resp. $\beta^{-1}(u)$) is called α -*fibre* (resp. β -*fibre*) of G at $u \in G_0$.

Convention. We write sometimes xy for $m(x, y)$, if $(x, y) \in G_{(2)}$. Whenever we write a product in a given groupoid, we are assuming that it is defined. \square

If $(G, \alpha, \beta; G_0)$ is a groupoid, the map $(\alpha, \beta) : G \rightarrow G_0 \times G_0$ defined by $(\alpha, \beta)(x) := (\alpha(x), \beta(x))$, $(\forall) x \in G$ is called the *anchor map* of G . If the anchor map $(\alpha, \beta) : G \rightarrow G_0 \times G_0$ is surjective, we say that (G, G_0) is *transitive*.

In the following proposition we summarize some basic rules of algebraic calculation in a Ehresmann groupoid obtained directly from definitions.

Proposition 2.1. ([4]) *In a Ehresmann groupoid $(G, \alpha, \beta, m, \varepsilon, i, G_0)$ the following assertions hold :*

- (i) $\alpha(xy) = \alpha(x)$ and $\beta(xy) = \beta(y)$ for any $(x, y) \in G_{(2)}$;
- (ii) $\alpha(x^{-1}) = \beta(x)$ and $\beta(x^{-1}) = \alpha(x)$, $\forall x \in G$;
- (iii) $\alpha(\varepsilon(u)) = u$ and $\beta(\varepsilon(u)) = u$, $(\forall) u \in G_0$;
- (iv) $\varepsilon(u) \cdot \varepsilon(u) = \varepsilon(u)$ and $(\varepsilon(u))^{-1} = \varepsilon(u)$ for each $u \in G_0$;
- (v) if $(x, y) \in G_{(2)}$, then $(y^{-1}, x^{-1}) \in G_{(2)}$ and $(x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$;
- (vi) for any $u \in G_0$, the set $G(u) := \alpha^{-1}(u) \cap \beta^{-1}(u)$ is a group under the restriction of the multiplication, called the **isotropy group at u** of G ;
- (vii) $\varphi : G(\alpha(x)) \rightarrow G(\beta(x))$, $\varphi(z) := x^{-1}zx$ is an isomorphism of groups.
- (viii) if (G, G_0) is transitive, then all isotropy groups are isomorphic.

Applying Proposition 2.1, it is easily to prove the following proposition.

Proposition 2.2. ([4]) *Let $(G, \alpha, \beta, m, \varepsilon, i, G_0)$ be a Ehresmann groupoid. The structure functions of G verifies the following relations:*

- (i) $\alpha \circ i = \beta$, $\beta \circ i = \alpha$, $i \circ \varepsilon = \varepsilon$ and $i \circ i = Id_G$;
- (ii) $\alpha \circ \varepsilon = \beta \circ \varepsilon = Id_{G_0}$.

Remark 2.1. *Let $(G, \alpha, \beta, m, \varepsilon, i, G_0)$ be a Ehresmann groupoid. If $G_0 \subseteq G$ and $\varepsilon = Id_{G_0}$, then $(G, \alpha, \beta, m, i, G_0)$ is a Brandt groupoid ([6]).*

Definition 2.2. ([5]) *Let (G, α, β, G_0) and $(G', \alpha', \beta', G'_0)$ be two groupoids.*

- (i) *A morphism of groupoids or groupoid morphism from G into G' is a pair of maps (f, f_0) , where $f : G \rightarrow G'$ and $f_0 : G_0 \rightarrow G'_0$ such that the following conditions hold:*

- (1) $\alpha' \circ f = f_0 \circ \alpha$, $\beta' \circ f = f_0 \circ \beta$;
- (2) $f(m(x, y)) = m'(f(x), f(y))$ for all $(x, y) \in G_{(2)}$.
- (ii) A groupoid morphism $(f, f_0) : (G, G_0) \rightarrow (G', G'_0)$ such that f and f_0 are bijective maps, is called **isomorphism of groupoids**.

Proposition 2.3. ([4]) If $(f, f_0) : (G, G_0) \rightarrow (G', G'_0)$ is groupoid morphism, then the following relations hold:

$$f \circ \varepsilon = \varepsilon' \circ f_0 \quad \text{and} \quad f \circ i = i' \circ f.$$

If $G_0 = G'_0$ and $f_0 = Id_{G_0}$, we say that $f : (G, G_0) \rightarrow (G', G_0)$ is a G_0 -morphism of groupoids.

Example 2.1.(i) A nonempty set G_0 may be considered to be a groupoid over G_0 , called the *nul groupoid* associated to G_0 . For this, we take $\alpha = \beta = \varepsilon = i = Id_{G_0}$ and $u \cdot u = u$ for all $u \in G_0$.

(ii) A group G having e as unity may be considered to be a $\{e\}$ -groupoid with respect to structure functions:

$$\alpha(x) = \beta(x) := e; \quad G_{(2)} := \{(x, y) \in G \times G \mid \beta(x) = \alpha(y)\} = G \times G, \quad m(x, y) := xy;$$

$$\varepsilon : \{e\} \rightarrow G, \quad \varepsilon(e) := e \quad \text{and} \quad i : G \rightarrow G, \quad i(x) := x^{-1}.$$

Conversely, a groupoid with one unit, i.e. $G_0 = \{e\}$, is a group.

(iii) The Cartesian product $G := X \times X$ has a structure of groupoid over X by taking the structure functions as follows: $\overline{\alpha}(x, y) := x$, $\overline{\beta}(x, y) := y$; the elements (x, y) and (y', z) are composable in $G := X \times X$ iff $y' = y$ and we define $(x, y) \cdot (y, z) = (x, z)$, the inclusion map $\overline{\varepsilon} : X \rightarrow X \times X$ is given by $\overline{\varepsilon}(x) := (x, x)$ and the inverse of (x, y) is defined by $(x, y)^{-1} := (y, x)$. This is called the *pair groupoid* associated to set X . Its unit set is $G_0 = \{(x, x) \in X \times X \mid x \in X\}$. \square

3 Generalized vector groupoids

In this section we introduce a structure of Ehresmann groupoid on a pair of vector spaces.

Definition 3.1. A generalized vector groupoid over a field K , is a Ehresmann groupoid $(V, \alpha, \beta, \odot, \varepsilon, i, V_0)$ such that:

(3.1.1) V and V_0 are vector spaces over K .

(3.1.2) The source and the target maps $\alpha : V \rightarrow V_0$ and $\beta : V \rightarrow V_0$ are linear maps.

(3.1.3) The inclusion $\varepsilon : V_0 \rightarrow V$ and the inversion $i : V \rightarrow V$, $x \mapsto i(x) := x^{-1}$ are linear maps and the following condition is verified:

$$(1) \quad x + x^{-1} = \varepsilon(\alpha(x)) + \varepsilon(\beta(x)), \quad \text{for all } x \in V.$$

(3.1.4) The map $m : V_{(2)} := \{(x, y) \in V \times V \mid \alpha(y) = \beta(x)\} \rightarrow V$, $(x, y) \mapsto m(x, y) := x \odot y$, satisfy the following conditions :

- (1) $x \odot (y + z - \varepsilon(\beta(x))) = x \odot y + x \odot z - x$, for all $x, y, z \in V$, such that $\alpha(y) = \beta(x) = \alpha(z)$.
- (2) $x \odot (ky + (1 - k)\varepsilon(\beta(x))) = k(x \odot y) + (1 - k)x$, for all $x, y \in V$, such that $\alpha(y) = \beta(x)$.
- (3) $(y + z - \varepsilon(\alpha(x))) \odot x = y \odot x + z \odot x - x$, for all $x, y, z \in V$, such that $\alpha(x) = \beta(y) = \beta(z)$.
- (4) $(ky + (1 - k)\varepsilon(\alpha(x))) \odot x = k(y \odot x) + (1 - k)x$ for all $x, y \in V$, such that $\alpha(x) = \beta(y)$.

When there can be no confusion we put xy or $x \cdot y$ instead of $x \odot y$.

From Definition 3.1 follows the following corollary.

Corollary 3.1. *Let $(V, \alpha, \beta, \odot, \iota, V_0)$ be a vector groupoid. Then:*

- (i) *The source and target $\alpha, \beta : V \rightarrow V_0$ are linear epimorphisms.*
- (ii) *The inversion $\iota : V \rightarrow V$ is a linear automorphism.*
- (iii) *The fibres $\alpha^{-1}(0)$ and $\beta^{-1}(0)$ and the isotropy group*

$V(0) := \alpha^{-1}(0) \cap \beta^{-1}(0)$ *are vector subspaces of the vector space V .*

Proposition 3.1. *If $(V, \alpha, \beta, \odot, \varepsilon, i, V_0)$ is a vector groupoid, then:*

- (i) $\varepsilon(0) \odot x = x, \forall x \in \alpha^{-1}(0);$
- (ii) $x \odot \varepsilon(0) = x, \forall x \in \beta^{-1}(0).$

Proof. (i) If $x \in \alpha^{-1}(0)$, then $\alpha(x) = 0$. We have $\beta(\varepsilon(0)) = 0$, since $\beta \circ \varepsilon = Id_{V_0}$. So $(\varepsilon(0), x) \in V_{(2)}$ and, using the condition (G2) from Definition 2.1, one obtains that $\varepsilon(0) \cdot x = \varepsilon(\alpha(x)) \cdot x = x$. Similarly, we prove that the assertion(ii) holds. \square

Remark 3.1. *If in Definition 3.1, we consider $V_0 \subseteq V$ and $\varepsilon = Id_{V_0}$, then $(V, \alpha, \beta, \odot, \varepsilon, i, V_0)$ is a vector groupoid, see [6]. In this case, we will say that (V, V_0) is a vector V_0 -groupoid.*

Example 3.1. Let V be a vector space over a field K . If we define the maps $\alpha_0, \beta_0, \iota_0 : V \rightarrow V$, $\alpha_0(x) = \beta_0(x) = 0$, $\iota_0(x) = -x$, and $m_0(x, y) = x + y$, then $(V, \alpha_0, \beta_0, m_0, \iota_0, V_0 = \{0\})$ is a vector groupoid called *vector groupoid with a single unit*. Therefore, each vector space V can be regarded as vector $\{0\}$ -groupoid. \square

Example 3.2. Let V be a vector space over a field K . Then V has a structure of null groupoid over V (see Example 2.1(i)). The structure functions are $\alpha = \beta = \varepsilon = \iota = Id_V$ and $x \odot x = x$ for all $x \in V$. We have that $V_0 = V$ and the maps $\alpha, \beta, \varepsilon, \iota$ are linear. Since $x + \iota(x) = x + x$ and $\alpha(x) + \beta(x) = x + x$ imply that the condition 3.1.3(1) holds. It is easy to verify the conditions 3.1.4(1)- 3.1.4(4) from Definition 3.1. Then V is a vector groupoid, called the *null vector groupoid* associated to V . \square

Example 3.3. Let V be a vector space over a field K . We consider the pair groupoid $(V \times V, \bar{\alpha}, \bar{\beta}, \bar{m}, \bar{\varepsilon}, \bar{i}, V)$ associated to V (see Example 2.1(iii)). We have that $V \times V$ is a vector space and the source $\bar{\alpha}$ and target $\bar{\beta}$ are linear maps. Also, the inclusion map $\bar{\varepsilon} : V \rightarrow V \times V$ and the inversion map $\bar{i} : V \times V \rightarrow V \times V$ are linear. For all $(x, y) \in V \times V$ we have

$$(x, y) + \bar{i}(x, y) = (x, y) + (y, x) = (x + y, x + y) \text{ and}$$

$$\bar{\varepsilon}(\bar{\alpha}(x, y)) + \bar{\varepsilon}(\bar{\beta}(x, y)) = \bar{\varepsilon}(x) + \bar{\varepsilon}(y) = (x, x) + (y, y) = (x + y, x + y)$$

and it follows that $(x, y) + \bar{i}(x, y) = \bar{\varepsilon}(\bar{\alpha}(x, y)) + \bar{\varepsilon}(\bar{\beta}(x, y))$.

Therefore, the conditions (3.1.1) – (3.1.3) are satisfied.

For to prove that the condition (3.1.4)(1) is verified, we consider the elements $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2)$ from $V \times V$ such that $\bar{\alpha}(y) = \bar{\beta}(x) = \bar{\alpha}(z)$. Then $y_1 = x_2 = z_1$. We have

$$\begin{aligned} x \odot_{V \times V} (y + z - \bar{\varepsilon}(\bar{\beta}(x))) &= (x_1, x_2) \odot_{V \times V} ((x_2, y_2) + (x_2, z_2) - \bar{\varepsilon}(\bar{\beta}(x_1, x_2))) = \\ &= (x_1, x_2) \odot_{V \times V} ((x_2, y_2) + (x_2, z_2) - \bar{\varepsilon}(x_2)) = (x_1, x_2) \odot_{V \times V} ((x_2, y_2) + (x_2, z_2) - \\ &- (x_2, x_2)) = (x_1, x_2) \odot_{V \times V} (x_2, y_2 + z_2 - x_2) = (x_1, y_2 + z_2 - x_2). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} x \odot_{V \times V} y + x \odot_{V \times V} z - x &= (x_1, x_2) \odot_{V \times V} (x_2, y_2) + (x_1, x_2) \odot_{V \times V} (x_2, z_2) - (x_1, x_2) = \\ &= (x_1, y_2) + (x_1, z_2) - (x_1, x_2) = (x_1, y_2 + z_2 - x_2). \end{aligned}$$

Hence, $x \odot_{V \times V} (y + z - \bar{\varepsilon}(\bar{\beta}(x))) = x \odot_{V \times V} y + x \odot_{V \times V} z - x$ and so the relation (3.1.4)(1) holds.

In the same manner, we prove that the relations 3.1.4(2) - 3.1.4(4) are verified. Hence $V \times V$ is a vector groupoid called the *pair vector groupoid* associated to V . \square

Definition 3.2. Let $(V_1, \alpha_1, \beta_1, \odot_1, \varepsilon_1, i_1, V_{1,0})$ and $(V_2, \alpha_2, \beta_2, \odot_2, \varepsilon_2, i_2, V_{2,0})$ be two vector groupoids. A groupoid morphism $(f, f_0) : (V_1, V_{1,0}) \rightarrow (V_2, V_{2,0})$ with property that $f : V_1 \rightarrow V_2$ and $f_0 : V_{1,0} \rightarrow V_{2,0}$ are linear maps, is called **vector groupoid morphism** or **morphism of vector groupoids**.

If $V_{2,0} = V_{1,0}$ and $f_0 = Id_{V_{1,0}}$, then we say that $(f, Id_{V_{1,0}}) : (V_1, V_{1,0}) \rightarrow (V_2, V_{1,0})$ is a $V_{1,0}$ - **morphism of vector groupoids**. It is denoted by $f : V_1 \rightarrow V_2$.

Proposition 3.2. Let $(V, \alpha, \beta, \odot, \varepsilon, i, V_0)$ be a vector groupoid. The anchor map $(\alpha, \beta) : V \rightarrow V_0 \times V_0$ is a V_0 - morphism of vector groupoids from the vector groupoid (V, V_0) into the pair vector groupoid $(V_0 \times V_0, \bar{\alpha}, \bar{\beta}, \bar{m}, \bar{\varepsilon}, \bar{i}, V_0)$.

Proof. We denote $(\alpha, \beta) := f$. Then $f(x) = (\alpha(x), \beta(x))$, for all $x \in V$. We prove that $\bar{\alpha} \circ f = Id_{V_0} \circ \alpha$ or equivalently, $\bar{\alpha} \circ f = \alpha$. Indeed, for all $x \in V$ we have $(\bar{\alpha} \circ f)(x) = \bar{\alpha}(f(x)) = \bar{\alpha}(\alpha(x), \beta(x)) = \alpha(x)$.

Therefore, $\bar{\alpha} \circ f = \alpha$. Similarly, we verify that $\bar{\beta} \circ f = \beta$. Hence, the condition (i) from Definition 2.2 is satisfied. For $(x, y) \in V_{(2)}$ we have $f(x \odot_V y) = (\alpha(x \odot_V y), \beta(x \odot_V y)) = (\alpha(x), \beta(y))$ and $\tilde{m}(f(x), f(y)) = \tilde{m}((\alpha(x), \beta(x)), (\alpha(y), \beta(y))) = (\alpha(x), \beta(y))$.

Therefore, $f(x \odot_V y) = \tilde{m}(f(x), f(y))$, for all $(x, y) \in V_{(2)}$. Hence, the equality (ii) from Definition 2.2 is verified. Thus $f : V \rightarrow V_0 \times V_0$ is a V_0 - morphism of groupoids.

Let $x, y \in V$ and $a, b \in K$. Since α, β are linear maps, we have
 $f(ax + by) = (\alpha(ax + by), \beta(ax + by)) = (a\alpha(x) + b\alpha(y), a\beta(x) + b\beta(y)) =$
 $= a(\alpha(x), \beta(x)) + b(\alpha(y), \beta(y)) = af(x) + bf(y),$
 i.e. f is a linear map. Therefore, the conditions from Definition 3.2 are verified.
 Hence f is a V_0 - morphism of vector groupoids. \square

4 The induced vector groupoid of a vector groupoid by a linear map

Let $(V, \alpha, \beta, \odot_V, \varepsilon, i, V_0)$ be a vector groupoid (in the sense of Definition 3.1) and let $h : X \rightarrow V_0$ be a linear map. We consider the set:

$$h^*(V) = \{(x, y, a) \in X \times X \times V \mid h(x) = \alpha(a), h(y) = \beta(a)\}.$$

Since X and V are vector spaces and h is a linear map, one obtains that $h^*(V)$ has a canonical structure of vector space.

For the pair $(h^*(V), X)$ we define the following structure functions.

The source and target $\alpha^*, \beta^* : h^*(V) \rightarrow X$, inversion $i^* : h^*(V) \rightarrow h^*(V)$ and inclusion $\varepsilon^* : X \rightarrow h^*(V)$ are defined by

$$\begin{aligned} \alpha^*(x, y, a) &:= x, \quad \beta^*(x, y, a) := y, \quad i^*(x, y, a) := (y, x, i(a)), \quad (\forall) (x, y, a) \in h^*(V), \\ \varepsilon^*(x) &:= (x, x, \varepsilon(h(x))), \quad (\forall) x \in X. \end{aligned}$$

The partially multiplication $\odot_{h^*(V)} : h^*(V)_{(2)} \rightarrow h^*(V)$, where $h^*(V)_{(2)} := \{((x, y, a), (y', z, b)) \in h^*(V) \times h^*(V) \mid y = y' \text{ and } (a, b) \in V_{(2)}\}$ is given by:

$$(x, y, a) \odot_{h^*(V)} (y, z, b) := (x, z, a \odot_V b).$$

Proposition 4.1. $(h^*(V), \alpha^*, \beta^*, \odot_{h^*(V)}, \varepsilon^*, i^*, X)$ is a vector groupoid, called the induced vector groupoid of (V, V_0) by the linear map $h : X \rightarrow V_0$.

Proof. (i) It must to verify that the conditions of Definition 2.1 are satisfied. For this, we consider an arbitrary element $x = (x_1, y_1, a_1) \in h^*(V)$. Then $h(x_1) = \alpha(a_1)$ and $h(y_1) = \beta(a_1)$.

(a) Let now $y, z \in h^*(V)$ such that $\beta^*(x) = \alpha^*(y)$ and $\beta^*(y) = \alpha^*(z)$. Then, if we take $y = (x_2, y_2, a_2), z = (x_3, y_3, a_3)$, it follows $y_1 = x_2$ and $y_2 = x_3$. Therefore, $y = (y_1, y_2, a_2)$ and $z = (y_2, y_3, a_3)$. We have

- (1) $(x \odot_{h^*(V)} y) \odot_{h^*(V)} z = ((x_1, y_1, a_1) \odot_{h^*(V)} (y_1, y_2, a_2)) \odot_{h^*(V)} (y_2, y_3, a_3) =$
 $= (x_1, y_2, a_1 \odot_V a_2) \odot_{h^*(V)} (y_2, y_3, a_3) = (x_1, y_3, (a_1 \odot_V a_2) \odot_V a_3)$ and
- (2) $x \odot_{h^*(V)} (y \odot_{h^*(V)} z) = (x_1, y_1, a_1) \odot_{h^*(V)} ((y_1, y_2, a_2) \odot_{h^*(V)} (y_2, y_3, a_3)) =$
 $= (x_1, y_1, a_1) \odot_{h^*(V)} (y_1, y_3, a_2 \odot_V a_3) = (x_1, y_3, a_1 \odot_V (a_2 \odot_V a_3)).$

Using the relations (1),(2) and the fact that \odot_V is associative in V , it follows $(x \odot_{h^*(V)} y) \odot_{h^*(V)} z = x \odot_{h^*(V)} (y \odot_{h^*(V)} z)$. Hence, the condition (G1) from Definition 2.1 holds.

(b) We have

$$\begin{aligned} \varepsilon^*(\alpha^*(x)) \odot_{h^*(V)} x &= \varepsilon^*(\alpha^*(x_1, y_1, a_1)) \odot_{h^*(V)} (x_1, y_1, a_1) = \varepsilon^*(x_1) \odot_{h^*(V)} (x_1, y_1, a_1) = \\ &= (x_1, x_1, \varepsilon(h(a_1))) \odot_{h^*(V)} (x_1, y_1, a_1) = (x_1, y_1, \varepsilon(h(a_1))) \odot_V a_1 = (x_1, y_1, a_1) = x. \end{aligned}$$

Similarly, we verify that $x \odot_{h^*(V)} \varepsilon^*(\beta^*(x)) = x$. Hence, the condition (G2) from Definition 2.1 holds.

(c) We have

$$(3) \quad i^*(x) \odot_{h^*(V)} x = i^*(x_1, y_1, a_1) \odot_{h^*(V)} (x_1, y_1, a_1) = (y_1, x_1, i(a_1)) \odot_{h^*(V)} (x_1, y_1, a_1) = \\ = (y_1, y_1, i(a_1) \odot_V a_1) = (y_1, y_1, \varepsilon(\beta(a_1))) \text{ and}$$

$$(4) \quad \varepsilon^*(\beta^*(x)) = \varepsilon^*(\beta^*(x_1, y_1, a_1)) = \varepsilon^*(y_1) = (y_1, y_1, \varepsilon(h(y_1))) = (y_1, y_1, \varepsilon(\beta(a_1))).$$

From the relations (3) and (4) it follows $i^*(x) \odot_{h^*(V)} x = \varepsilon^*(\beta^*(x))$.

Similarly, we prove that $x \odot_{h^*(V)} i^*(x) = \varepsilon^*(\alpha^*(x))$. Hence, the condition (G3) from Definition 2.1 holds. Then $(h^*(V), \alpha^*, \beta^*, \odot_{h^*(V)}, \varepsilon^*, i^*, X)$ is a groupoid.

(ii) We prove that the conditions from Definition 3.1 are verified.

It is clearly that the condition (3.1.1) from Definition 3.1 is verified.

Let now two elements $x, y \in h^*(V)$ and $k_1, k_2 \in K$, where $x = (x_1, y_1, a_1)$ and $y = (x_2, y_2, a_2)$. We have

$$\begin{aligned} \alpha^*(k_1 x + k_2 y) &= \alpha^*(k_1 x_1 + k_2 x_2, k_1 y_1 + k_2 y_2, k_1 a_1 + k_2 a_2) = \\ &= k_1 x_1 + k_2 x_2 = k_1 \alpha^*(x_1, y_1, a_1) + k_2 \alpha^*(x_2, y_2, a_2) = k_1 \alpha^*(x) + k_2 \alpha^*(y). \end{aligned}$$

It follows that α^* is a linear map. Similarly we prove that β^* is a linear map. Therefore the conditions (3.1.2) from Definition 3.1 hold.

Let now two elements $x_1, x_2 \in X$ and $k_1, k_2 \in K$. Using the fact that h is a linear map, one obtains that ε^* is linear. Indeed, we have

$$\begin{aligned} \varepsilon^*(k_1 x_1 + k_2 x_2) &= (k_1 x_1 + k_2 x_2, k_1 x_1 + k_2 x_2, h(k_1 x_1 + k_2 x_2)) = \\ &= (k_1 x_1 + k_2 x_2, k_1 x_1 + k_2 x_2, k_1 h(x_1) + k_2 h(x_2)) = k_1 (x_1, x_1, h(x_1)) + k_2 (x_2, x_2, h(x_2)) = \\ &= k_1 \varepsilon^*(x_1) + k_2 \varepsilon^*(x_2). \end{aligned}$$

Let $x, y \in h^*(V)$ and $k_1, k_2 \in K$, where $x = (x_1, y_1, a_1)$ and $y = (x_2, y_2, a_2)$. Using the linearity of the map i , one obtains that i^* is linear. Indeed, we have

$$\begin{aligned} i^*(k_1 x + k_2 y) &= i^*(k_1 x_1 + k_2 x_2, k_1 y_1 + k_2 y_2, k_1 a_1 + k_2 a_2) = \\ &= (k_1 y_1 + k_2 y_2, k_1 x_1 + k_2 x_2, i(k_1 a_1 + k_2 a_2)) = (k_1 y_1 + k_2 y_2, k_1 x_1 + k_2 x_2, k_1 i(a_1) + \\ &+ k_2 i(a_2)) = k_1 (y_1, x_1, i(a_1)) + k_2 (y_2, x_2, i(a_2)) = k_1 i^*(x_1, y_1, a_1) + k_2 i^*(x_2, y_2, a_2) = \\ &= k_1 i^*(x) + k_2 i^*(y). \end{aligned}$$

Let now $(x, y, a) \in h^*(V)$. Then $h(x) = \alpha(a)$ and $h(y) = \beta(a)$. We have

$$\begin{aligned} (x, y, a) + i^*(x, y, a) &= (x, y, a) + (y, x, i(a)) = (x + y, y + x, a + i(a)), \text{ and} \\ \varepsilon^*(\alpha^*(x, y, a)) + \varepsilon^*(\beta^*(x, y, a)) &= \varepsilon^*(x) + \varepsilon^*(y) = (x, x, \varepsilon(h(x))) + (y, y, \varepsilon(h(y))) = \end{aligned}$$

$$\begin{aligned}
&= (x + y, x + y, \varepsilon(h(x)) + \varepsilon(h(y))) = (x + y, x + y, \varepsilon(\alpha(a)) + \varepsilon(\beta(a))) = \\
&= (x + y, y + x, a + i(a)).
\end{aligned}$$

It follows $(x, y, a) + i^*(x, y, a) = \varepsilon^*(\alpha^*(x, y, a)) + \varepsilon^*(\beta^*(x, y, a))$, and so the condition (3.1.3)(1) from Definition 3.1 holds.

For to verify the relation 3.1.4(1) from Definition 3.1 we consider the arbitrary elements $x, y, z \in h^*(V)$, where $x = (x_1, y_1, a_1)$, $y = (x_2, y_2, a_2)$ and $z = (x_3, y_3, a_3)$ such that $\alpha^*(y) = \beta^*(x) = \alpha^*(z)$. Then $x_2 = y_1 = x_3$ and follows $x = (x_1, y_1, a_1)$, $y = (y_1, y_2, a_2)$ and $z = (y_1, y_3, a_3)$ with $a_1, a_2, a_3 \in V$ such that $h(x_j) = \alpha(a_j)$ and $h(y_j) = \beta(a_j)$ for $j = 1, 2, 3$.

$$\begin{aligned}
(5) \quad &x \odot_{h^*(V)} (y + z - \varepsilon^*(\beta^*(x))) = (x_1, y_1, a_1) \odot_{h^*(V)} ((y_1, y_2, a_2) + \\
&+ (y_1, y_3, a_3) - \varepsilon^*(\beta^*(x_1, y_1, a_1))) = (x_1, y_1, a_1) \odot_{h^*(V)} ((y_1, y_2, a_2) + (y_1, y_3, a_3) - \\
&\varepsilon^*(y_1)) = (x_1, y_1, a_1) \odot_{h^*(V)} ((y_1, y_2, a_2) + (y_1, y_3, a_3) - (y_1, y_1, \varepsilon(h(y_1)))) = \\
&= (x_1, y_1, a_1) \odot_{h^*(V)} (y_1, y_2 + y_3 - y_1, a_2 + a_3 - \varepsilon(\beta(a_1))) = \\
&= (x_1, y_2 + y_3 - y_1, a_1 \odot_V (a_2 + a_3 - \varepsilon(\beta(a_1)))) \text{ and}
\end{aligned}$$

$$\begin{aligned}
(6) \quad &x \odot_{h^*(V)} y + x \odot_{h^*(V)} z - x = (x_1, y_1, a_1) \odot_{h^*(V)} (y_1, y_2, a_2) + \\
&+ (x_1, y_1, a_1) \odot_{h^*(V)} (y_1, y_3, a_3) - (x_1, y_1, a_1) = (x_1, y_2, a_1 \odot_V a_2) + \\
&+ (x_1, y_3, a_1 \odot_V a_2) - (x_1, y_1, a_1) = (x_1, y_2 + y_3 - y_1, a_1 \odot_V a_2 + a_1 \odot_V a_3 - a_1).
\end{aligned}$$

Using (5), (6) and the hypothesis

$$\begin{aligned}
&a_1 \odot_V (a_2 + a_3 - \varepsilon(\beta(a_1))) = a_1 \odot_V a_2 + a_1 \odot_V a_3 - a_1, \text{ one obtains} \\
&x \odot_{h^*(V)} (y + z - \varepsilon^*(\beta^*(x))) = x \odot_{h^*(V)} y + x \odot_{h^*(V)} z - x.
\end{aligned}$$

Hence the condition 3.1.4(1) from Definition 3.1 holds.

Let now $x = (x_1, y_1, a_1) \in h^*(V)$, $y = (y_1, y_2, a_2) \in h^*(V)$ such that $h(x_1) = \alpha(a_1)$, $h(y_1) = \alpha(a_2) = \beta(a_1)$ and $h(y_2) = \beta(a_2)$. For all $k \in K$, we have

$$\begin{aligned}
(7) \quad &x \odot_{h^*(V)} (ky + (1 - k)\varepsilon^*(\beta^*(x))) = (x_1, y_1, a_1) \odot_{h^*(V)} (k(y_1, y_2, a_2) + \\
&+ (1 - k)\varepsilon^*(\beta^*(x_1, y_1, a_1))) = (x_1, y_1, a_1) \odot_{h^*(V)} (k(y_1, y_2, a_2) + (1 - k)\varepsilon^*(y_1)) = \\
&= (x_1, y_1, a_1) \odot_{h^*(V)} (k(y_1, y_2, a_2) + (1 - k)\varepsilon(h(y_1))) = (x_1, y_1, a_1) \odot_{h^*(V)} ((ky_1, ky_2, ka_2) + \\
&+ (1 - k)(y_1, y_1, \varepsilon(\beta(a_1)))) = (x_1, y_1, a_1) \odot_{h^*(V)} (y_1, ky_2 + (1 - k)y_1, ka_2 + \varepsilon(\beta(a_1))) = \\
&= (x_1, ky_2 + (1 - k)y_1, a_1 \odot_V (ka_2 + (1 - k)\varepsilon(\beta(a_1)))) \text{ and}
\end{aligned}$$

$$\begin{aligned}
(8) \quad &k(x \odot_{h^*(V)} y) + (1 - k)x = k((x_1, y_1, a_1) \odot_{h^*(V)} (y_1, y_2, a_2)) + \\
&+ (1 - k)(x_1, y_1, a_1) = k(x_1, y_2, a_1 \odot_V a_2) + (1 - k)(x_1, y_1, a_1) = \\
&= (x_1, ky_2 + (1 - k)y_1, k(a_1 \odot_V a_2) + (1 - k)a_1).
\end{aligned}$$

Using the equalities (7) and (8) and the hypothesis

$$a_1 \odot_V (ka_2 + (1 - k)\varepsilon(\beta(a_1))) = k(a_1 \odot_V a_2) + (1 - k)a_1,$$

one obtains that the condition 3.1.4(2) from Definition 3.1 holds.

In the same manner we prove that the conditions 3.1.4 (3) and 3.1.4 (4) hold. Hence $(h^*(V), \alpha^*, \beta^*, \odot_{h^*(V)}, \varepsilon^*, i^*, X)$ is a vector groupoid \square

Proposition 4.2. *The pair $(h_V^*, h) : (h^*(V), X) \longrightarrow (V, V_0)$ is a vector groupoid morphism, called the **canonical vector groupoid morphism on $h^*(V)$** , where*

the map $h_V^* : h^*(V) \longrightarrow V$ is defined by:

$$h_V^*(x, y, a) := a, \quad (\forall) (x, y, a) \in h^*(V).$$

Proof. Let $(x, y, a) \in h^*(V)$. Then $h(x) = \alpha(a)$ and $h(y) = \beta(a)$. We have

$$(\alpha \circ h_V^*)(x, y, a) = \alpha(h_V^*(x, y, a)) = \alpha(a) = h(x) = h(\alpha^*(x, y, a)) = (h \circ \alpha^*)(x, y, a).$$

Therefore, $\alpha \circ h_V^* = h \circ \alpha^*$. Similarly, we prove that $\beta \circ h_V^* = h \circ \beta^*$.

For all $(x_1, y_1, a_1), (y_1, y_2, a_2) \in h^*(V)$, we have

$$\begin{aligned} h_V^*((x_1, y_1, a_1) \odot_{h^*(V)} (y_1, y_2, a_2)) &= h_V^*(x_1, y_2, a_1 \odot_V a_2) = a_1 \odot_V a_2 = \\ &= h_V^*(x_1, y_1, a_1) \odot_V h_V^*(y_1, y_2, a_2). \end{aligned}$$

It is easy to prove that the map h_V^* is linear. Hence, the conditions from Definition 3.2 are verified. Therefore (h_V^*, h) is a morphism of vector groupoids. \square

Theorem 4.1. *The canonical vector groupoid morphism $(h_V^*, h) : (h^*(V), X) \longrightarrow (V, V_0)$ verify the universal property:*

(PU) : for every vector groupoid $(V', \alpha', \beta', \odot_{V'}, \varepsilon', i', X)$ and for every vector groupoid morphism $(u, h) : (V', X) \longrightarrow (V, V_0)$ there exists an unique X - morphism of vector groupoids $v : V' \longrightarrow h^(V)$ such that the diagram:*

$$\begin{array}{ccc} V' & \xrightarrow{u} & V \\ (\exists) v \searrow & & \nearrow h_V^* \\ & h^*(V) & \end{array}$$

is comutative, i.e. $h_V^ \circ v = u$.*

Proof. Define $v : V' \rightarrow h^*(V)$ by $v(a') := (\alpha'(a'), \beta'(a'), u(a'))$, $(\forall) a' \in V'$.

We have $v(a') \in h^*(V)$, since $h(\alpha'(a')) = \alpha(u(a'))$ and $h(\beta'(a')) = \beta(u(a'))$.

We have $\alpha^* \circ v = \alpha'$, since $(\alpha^* \circ v)(a') = \alpha^*(v(a')) = \alpha^*(\alpha'(a'), \beta'(a'), u(a')) = \alpha'(a')$. Similarly, we verify that $\beta^* \circ v = \beta'$.

Let $(a', b') \in V'_{(2)}$. Then $\beta'(a') = \alpha'(b')$. We have

$$v(a' \odot_{V'} b') = (\alpha'(a' \odot_{V'} b'), \beta'(a' \odot_{V'} b'), u(a' \odot_{V'} b')) = (\alpha'(a'), \beta'(b'), u(a') \odot_V u(b'))$$

and

$$\begin{aligned} v(a') \odot_{h^*(V)} v(b') &= (\alpha'(a'), \beta'(a'), u(a')) \odot_{h^*(V)} (\beta'(a'), \beta'(b'), u(b')) = \\ &= (\alpha'(a'), \beta'(b'), u(a') \odot_V u(b')). \end{aligned}$$

It follows that $v(a' \odot_{V'} b') = v(a') \odot_{h^*(V)} v(b')$.

Using the linearity of the maps α', β' and u , it is easy to verify that v is a linear map. Therefore, v is a X - morphism of vector groupoids.

We have $(h_V^* \circ v)(a') = h_V^*(v(a')) = h_V^*(\alpha'(a'), \beta'(a'), u(a')) = u(a')$, for all $a' \in V'$, i.e. $h_V^* \circ v = u$. Finally, we prove by a standard manner that v is unique. \square

Proposition 4.3. *The induced vector groupoid $(h^*(V), X)$ of a transitive vector groupoid (V, V_0) via the linear map $h : X \longrightarrow V_0$ is transitive.*

Proof. We prove that the anchor $(\alpha^*, \beta^*) : h^*(V) \rightarrow X \times X$ is a surjective map. For this, let an arbitrary element $(x, y) \in X \times X$. We have $(h(x), h(y)) \in V_0 \times V_0$. Since the anchor $(\alpha, \beta) : V \rightarrow V_0 \times V_0$ of the groupoid V is surjective, there exists an element $a \in V$ such that $(\alpha, \beta)(a) = (h(x), h(y))$. Then $\alpha(a) = h(x)$, $\beta(a) = h(y)$ and $(x, y, a) \in h^*(V)$. We have that $((\alpha^*, \beta^*)(x, y, a) = (\alpha^*(x, y, a), \beta^*(x, y, a)) = (x, y)$. Consequently, (α^*, β^*) is surjective. Hence $h^*(V)$ is a transitive vector groupoid. \square

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